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# Elementary proof of $\varepsilon \to +0$ limit of renormalised Feynman amplitudes†

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**Abstract.** An elementary proof of the distributional $\varepsilon \to +0$ limit of renormalised (absolutely convergent for $\varepsilon > 0$ in momentum space) Feynman amplitudes is given which makes use of simple and direct means and establishes the convergence of the amplitudes in Minkowski space.

## 1. Introduction

Various proofs of the distributional $\varepsilon \to +0$ limit of renormalised Feynman amplitudes have been given in the literature (e.g. Hepp 1966, Hahn and Zimmermann 1968, Zimmermann 1968, Lowenstein and Speer 1976) which rely heavily on already existing sophisticated mathematical results and are quite complex in nature. In this communication we provide a direct proof of the existence of the $\varepsilon \to +0$ limit of renormalised Feynman amplitudes, with the latter absolutely convergent for $\varepsilon > 0$ in momentum space (Hahn and Zimmermann 1968, Zimmermann 1968), which makes use of elementary tools, avoids the use of the more complicated theorems from distribution theory and is self contained. The derived result is of great importance as it provides the convergence, in the distributional sense, of the amplitudes in Minkowski space. For simplicity we consider only non-zero mass particles.

## 2. Proof of $\varepsilon \to +0$ limit of absolutely convergent Feynman amplitudes

A renormalised Feynman amplitude, in momentum space, may be written in the familiar form as

$$F_\varepsilon(P, \mu) = \int_{\mathbb{R}^{4n}} \mathrm{d}K\, A(P, K, \mu, \varepsilon) \prod_{l=1}^{L} D_l^{-1}, \qquad \varepsilon > 0, \tag{1}$$

where

$$D_l = [Q_l^2 + \mu_l^2 - \mathrm{i}\varepsilon(\boldsymbol{Q}_l^2 + \mu_l^2)], \tag{2}$$

$Q_l = \sum_{i=1}^{m} a_{li} p_i + \sum_{i=1}^{n} b_{li} k_i$, $A$ is a polynomial in its arguments and, in general, in the $\mu_j^{-1}$ as well. The latter is well known for theories with higher spin fields. We assume that $F_\varepsilon(P, \mu)$ is absolutely convergent for $\varepsilon > 0$. All that is required for the validity

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of our proof is that $\mu_j > 0$ for all $j$, and that the Feynman integral $F_\varepsilon(P, \mu)$ is *absolutely integrable* for $\varepsilon > 0$. $F_\varepsilon(P, \mu)$ may also be rewritten (Hahn and Zimmermann 1968, Zimmermann 1968, Lowenstein and Speer 1976, Manoukian 1983) in terms of Feynman parameters as

$$F_\varepsilon(P, \mu) = \int_D \mathrm{d}\alpha\, N(\alpha, P, \mu, \varepsilon)[G_\varepsilon(\alpha, P)]^{-t}, \tag{3}$$

where

$$G_\varepsilon(\alpha, P) = pUp + M^2 - \mathrm{i}\varepsilon(\boldsymbol{p} \cdot Up + M^2), \tag{4}$$

$$M^2 = \sum_{l=1}^{L} \alpha_l \mu_l^2, \tag{5}$$

and the matrix $U$ is rational in $\alpha$ and continuous almost everywhere in $D$, and it may be extended to a function continuous everywhere in $D$, where $D = \{\alpha = (\alpha_1, \ldots, \alpha_L): \alpha_i \geqslant 0, \Sigma_{i=1}^{L} \alpha_i = 1\} \cdot N(\alpha, P, \mu, \varepsilon)$ is rational in $\alpha$, and is a polynomial in the elements in $P$, $\mu$, in $\varepsilon$ and, in general, in the $\mu_j^{-1}$ as well. $t$ is some positive integer.

Let $f(P) \in \mathscr{S}(\mathbb{R}^{4m})$, and let $\chi(x)$ be a $C^\infty$ function such that $0 \leqslant \chi(x) \leqslant 1$, with $\chi(x) = 1$ for $-\frac{1}{3} \leqslant x$ and $\chi(x) = 0$ for $x < -\frac{2}{3}$ (Lowenstein and Speer 1976, Manoukian 1983). We set $x = pUp/\boldsymbol{\mu}^2$ and $\boldsymbol{\mu} = \min_j \mu_j$. We are interested in the limit of

$$T_\varepsilon(f) = \int_{\mathbb{R}^{4m}} \mathrm{d}P f(P) F_\varepsilon(P, \mu) \tag{6}$$

for $\varepsilon \to +0$. To this end we may rewrite (Lowenstein and Speer 1976, Manoukian 1983) (6) as $T_\varepsilon(f) = T_\varepsilon^1(f) + T_\varepsilon^2(f)$, where

$$T_\varepsilon^1(f) = \int_{\mathbb{R}^{4m}} \mathrm{d}P f(P) \int_D \mathrm{d}\alpha\, \chi(pUp/\boldsymbol{\mu}^2) N(\alpha, P, \mu, \varepsilon)[G_\varepsilon(\alpha, P)]^{-t}, \tag{7}$$

$$T_\varepsilon^2(f) = \int_{\mathbb{R}^{4m}} \mathrm{d}P f(P) \int_D \mathrm{d}\alpha\, [1 - \chi(pUp/\boldsymbol{\mu}^2)] N(\alpha, P, \mu, \varepsilon)[G_\varepsilon(\alpha, P)]^{-t}, \tag{8}$$

and the interchange of the order of integration is permitted since $F_\varepsilon(P, \mu)$ in (1) is assumed to be absolutely convergent. Because of the property of the function $\chi(x)$, $|G_\varepsilon(\alpha, P)|^{-1} \leqslant 3/\boldsymbol{\mu}^2$ in (7). Upon writing

$$N(\alpha, P, \mu, \varepsilon) = \sum_{a, m'} \varepsilon^a p^{m'} N_{am'}(\alpha, \mu)$$

where (Lowenstein and Speer 1976, Manoukian 1983)

$$\int_D \mathrm{d}\alpha\, |N_{am'}(\alpha, \mu)| < \infty, \tag{9}$$

the existence of the $\varepsilon \to +0$ limit of $T_\varepsilon^1(f)$ in (7) immediately follows by an elementary application of the Lebesgue dominated convergence theorem. The *difficulty* in establishing the $\varepsilon \to +0$ limit of $T_\varepsilon(f)$ is due not to $T_\varepsilon^1(f)$ but to $T_\varepsilon^2(f)$ which we now turn to. To this end we use the following elementary identities:

$$[G_\varepsilon(\alpha, P)]^{-t} = -(\boldsymbol{p} \cdot Up + M^2)^2 t(t+1) \int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda\, [G_\lambda(\alpha, P)]^{-t-2}$$
$$+ (\boldsymbol{p} \cdot Up + M^2)\mathrm{i}t(\varepsilon - 1)[G_1(\alpha, P)]^{-t-1} + [G_1(\alpha, P)]^{-t}, \tag{10}$$

$$[G_\lambda(\alpha, P)]^{-t-2} = \frac{(-\frac{1}{2})^{t+1}}{(t+1)!}\left((pUp)_\lambda^{-1} \sum_{j=1}^{m} p_j^\rho \frac{\partial}{\partial p_j^\rho}\right)^{t+1} [G_\lambda(\alpha, P)]^{-1}, \tag{11}$$

$(pUp)_\lambda \equiv pUp - \mathrm{i}\lambda \boldsymbol{p} \cdot U\boldsymbol{p}$. The second and third terms in (10) when substituted in turn for $[G_\varepsilon(\alpha, P)]^{-t}$ in (8) cause no problems, and the $\varepsilon \to +0$ limit may be trivially taken since due to the continuity of $U$ in $\alpha$ we may bound $\boldsymbol{p} \cdot U\boldsymbol{p} + M^2$ by some polynomial in $p$. The only term which may cause problems in (10) is the first one. We note from the vanishing property of $f(P)$ together with all its derivatives at infinity, and upon integration by parts in (8), by using in the process the identity (11), that when the first term on the right-hand side of (10) is substituted for $[G_\varepsilon(\alpha, P)]^{-t}$ in $T_\varepsilon^2(f)$, the latter may be written as a finite sum of terms of the following form with expansion coefficients which have well defined limits for $\varepsilon \to +0$:

$$\int_{\mathbb{R}^{4m}} \mathrm{d}P \int_D \mathrm{d}\alpha\, N_{am'}(\alpha, \mu) \int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda \frac{[G_\lambda(\alpha, P)]^{-1}}{[(pUp)_\lambda]^{t+1}} h(P)\chi(\alpha, P). \tag{12}$$

Here we remark that $h(P) \in \mathcal{S}(\mathbb{R}^{4m})$, and $\chi(\alpha, P)$ is bounded and vanishes for $pUp > -\frac{1}{3}\boldsymbol{\mu}^2$. According to the latter constraint, $|(pUp)_\lambda| \geq |pUp| \geq \frac{1}{3}\boldsymbol{\mu}^2$ in (12). Finally we note that

$$|G_\lambda(\alpha, P)^{-1}| \leq 1/\lambda M^2 \leq c/\lambda,$$

where $c$ is some positive constant. Using the facts that $h(P) \in \mathcal{S}(\mathbb{R}^{4m})$, $\chi(\alpha, P)$ is bounded and the property in (9), the existence of the $\varepsilon \to +0$ limit of the expression in (12) then immediately follows from the Lebesgue dominated convergence theorem because of the following basic and very useful inequality:

$$\left|\int_\varepsilon^1 \mathrm{d}\lambda_1 \int_{\lambda_1}^1 \mathrm{d}\lambda \frac{[G_\lambda(\alpha, P)]^{-1}}{[(pUp)_\lambda]^{t+1}}\right| \leq \frac{(3c)^{t+2}}{3}\left|\int_\varepsilon^1 \mathrm{d}\lambda_1 \ln \lambda_1\right| \leq A, \qquad 0 \leq \varepsilon \leq 1,$$

where $A$ is some constant *independent* of $\varepsilon$. This completes the proof of the existence of the $\varepsilon \to +0$ limit of $T_\varepsilon(f)$ in (6).

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